Faster Exploration Of Some Temporal Graphs

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Overview

In this talk we present results on the exploration of 3 graph classes within the temporal setting:

- Temporal graphs for which the underlying graph is a Cycle with k-Chords can be explored in $O(kn)$ days.
- Temporal graphs for which the underlying graph has a (r, b)-division can be explored in $O\left((n + \max\{r, b\}(r + b))\frac{nb}{r}\log n\right)$ days. This can be applied to show that:
  - Temporal graphs for which the underlying graph has treewidth of k can be explored in $O(kn^{1.5}\log n)$ days.
  - Temporal graphs for which the underlying graph is planar can be explored in $O(n^{1.75}\log n)$ days.
- There exists temporal graphs for which the underlying graph is a sub cubic planar graph that can not be explored in faster than $\Omega(n\log n)$ days.

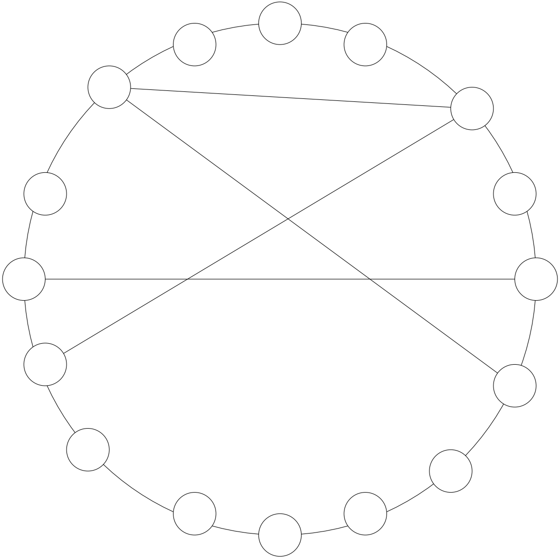
Assumptions

- The agent may start the exploration from any vertex in the graph.
- At each day, the agent may move along a single edge incident to the vertex it is currently on.
- The exploration is complete when the agent has visited every vertex in the graph.
- The graph is fully connected in every day.

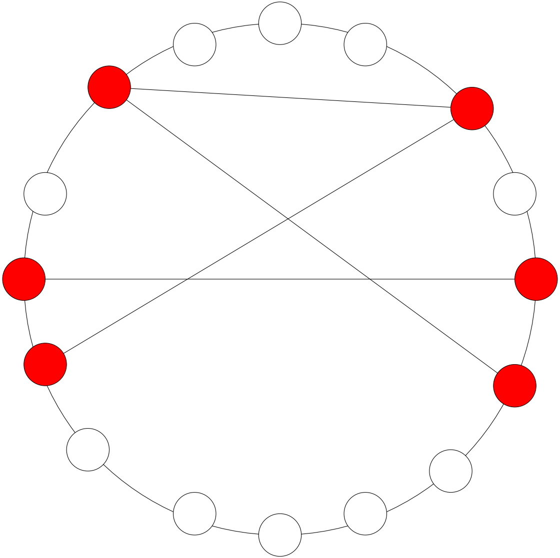
Cycles with k-Chords

- We show that any temporal realisation of a graph G that is a cycle with k-chords can be explored in $O(kn)$ days.
- Our results for exploring graphs with k-chords are based on the technical Lemma 3.
- At a (very) high level, we show that the path between pairs of vertices of degree ≥ 3 can always be explored in $O(n)$ days.
- As there are k chords, the graph can be partitioned into at most $2k$ paths between the vertices incident to each chord.
- Therefore, the whole graph can be explored in $O(kn)$ days.

Cycles with k-chords



Cycles with k-chords

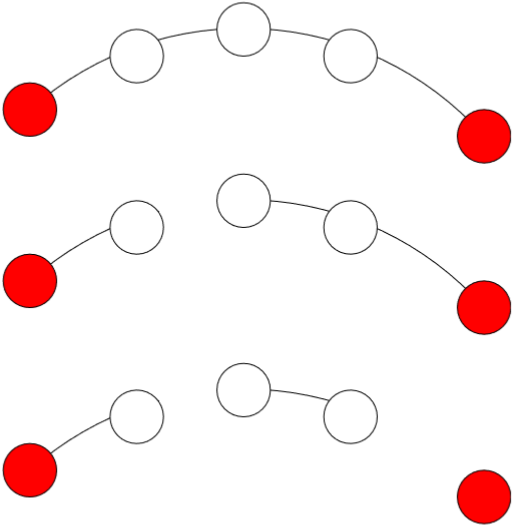


Cycles with k Chords

Lemma 3

Let $\mathcal{G} = (G_1, G_2, \ldots, G_{2n})$ be an n-vertex temporal graph of lifetime $T = 2n$, and let G be the underlying graph of $\mathcal{G}$. Let $P = (v_1, v_2, \ldots, v_\rho)$, $\rho \geq 1$, be a path in G such that every vertex of P, except possibly its endpoints v_1 and v_ρ, has degree 2 in G. Moreover, in every snapshot of $\mathcal{G}$ at most one edge of P is absent. Then there exists a vertex $v \in V(G)$ such that all vertices of P can be explored starting from v.

Intuition behind Lemma 3

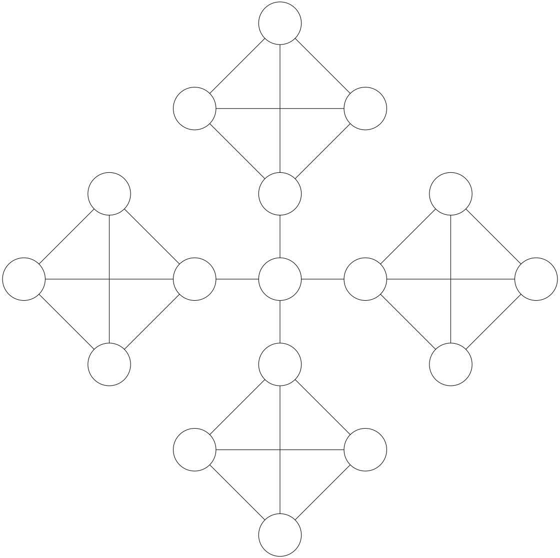


(r, b)-divisions

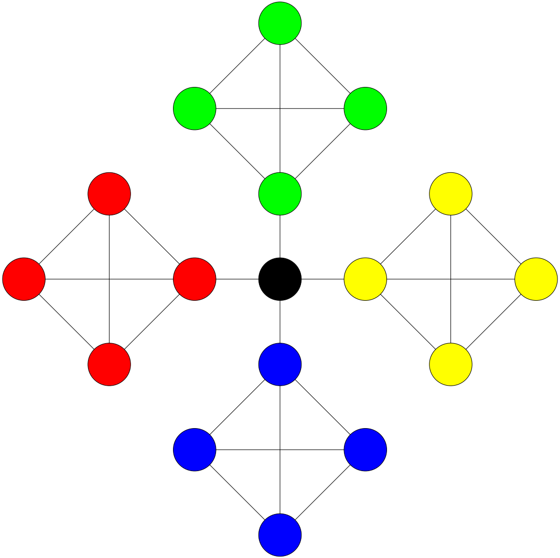
Our second set of results use graphs with an underlying (r, b) division. A graph $G = (V, E)$ has an (r, b) division if there exists a set $S \subseteq V$ and a partition of $G[V \setminus S]$ into $O(n/r)$ components, each associated with a **boundary set** of vertices from S such that:

1. Each component within the partition contains at most r vertices.
2. The boundary set of each component has at most b vertices.
3. The union of the boundary sets is S.
4. Every edge in G that has only one endpoint within a component has the other endpoint in the corresponding boundary set.

(r, b)-division example



$(4, 1)$-division example



Exploring with (r, b)-divisions

Theorem 7

A temporal graph $\mathcal{G}$, whose underlying graph has a (r, b)-division, can be explored in $O\left((n + \max\{r, b\}(r + b))\frac{nb}{r}\log n\right)$ days.

Intuition for Theorem 7

- The main idea is to use a set of b agents based in the boundary set.
- The goal is to show that each component can be explored in $O(\max\{r, b\}(r + b))$ days.
- As there are $\frac{n}{r}$ components to explore, and we need at most n days to move from one component to the next, we can explore the full graph in $O((n + \max\{r, b\}(r + b))\frac{n}{r})$ days with b agents.
- Using the multi-agent to single-agent Lemma due to Erlebach et. al. gives an upper bound on the exploration with a single agent of $O((n + \max\{r, b\}(r + b))\frac{nb}{r} \log n)$ [1].

Graphs with a treewidth of k

Lemma 8 (Adaptation of Lemma 4.4 in [1])

Any graph of treewidth at most k admits a $\left(2\sqrt{n}, 6k\right)$-division.

Theorem 9

An n-vertex temporal graph, whose underlying graph has treewidth at most k, can be explored in $O(kn^{1.5}\log n)$ days.

Planar Graphs

- Frederickson proved that all planar graphs admit a $(r, O(\sqrt{r}))$-division for any $1 \leq r \leq n$ [2].
- We can apply Theorem 7 with this division for $r = \sqrt{n}$:

Theorem 10

An n-vertex temporal graph, whose underlying graph is planar, can be explored in $O(n^{1.75} \log n)$ days.

Subcubic planar graphs

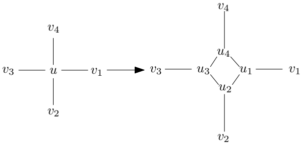
- Erlebach et. al. [1] show that there exists temporal realisations of n-vertex planar graphs with maximum degree 4 which can not be explored in fewer than $\Omega(n \log n)$ days.
- We strengthen this to show that there exists temporal realisations of planar graphs with maximum degree 3 that cannot be explored in fewer than $\Omega(n \log n)$ days.

Theorem 13

There exist temporal realisations of n-vertex subcubic planar graphs that cannot be explored faster than $\Omega(n \log n)$ days.

Intuition

- Our result relies on the edge contraction Lemma of Erlebach et. al. [1], which states that given a graph G such that every temporal realisation of G can be explored in t days, any graph G' that can be constructed by contracting edges in G can also be explored in t days.
- Taking as input some graph G that can not be explored in fewer than $\Omega(n \log n)$ days, we construct a graph G' with maximum degree 3 such that G is the contraction of G'.
- Using the edge contraction Lemma, G' can not be explored in fewer than $\Omega(n \log n)$ days without there being a way of exploring G in fewer than $\Omega(n \log n)$ days.



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